Satisfiability and Canonisation of Timely Constraints

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Abstract

We abstractly formulate an analytic problem that arises naturally in the study of coordination in multi-agent systems. Let I be a set of arbitrary cardinality (the set of *actions*) and assume that for each pair of distinct actions (i, j), we are given a number $\delta(i, j)$. We say that a function t, specifying a time for each action, *satisfies* the *timely constraint* δ if for every pair of distinct actions (i, j), we have $t(j) - t(i) \le \delta(i, j)$ (and thus also $t(j) - t(i) \ge -\delta(j, i)$). While the approach that first comes to mind for analysing these definitions is an analytic/geometric one, it turns out that graph-theoretic tools yield powerful results when applied to these definitions. Using such tools, we characterise the set of satisfiable timely constraints, and reduce the problem of satisfiability of a timely constraint to the all-pairs shortest-path problem, and for finite I, furthermore to the negative-cycle detection problem. Moreover, we constructively show that every satisfiable timely constraint has a *minimal* satisfying function — a key milestone on the way to optimally solving a large class of coordination problems — and reduce the problem of finding this minimal satisfying function, as well as the problems of classifying and comparing timely constraints, to the all-pairs shortest-path problem. At the heart of our analysis lies the constructive definition of a "nicely-behaved" representative $\hat{\delta}$ for each class of timely constraints sharing the same set of satisfying functions. We show that this canonical representative, as well as the map from such canonical representatives to the the sets of functions satisfying the classes of timely constraints they represent, has many desired properties, which provide deep insights into the structure underlying the above definitions.

Keywords: graph theory, distributed coordination, temporal coordination, real-time constraints, real-time system specification, multi-agent systems

1. Motivation and Definitions

In a distributed algorithm, multiple processes, or agents, work toward a common goal. More often than not, the actions of some agents are dependent on the previous execution (if not also on the outcome) of the actions of other agents. This, in turn, results in interdependencies between the timings of the actions of the various agents. In this note, we analyse such timing constraints

in an abstract setting, and characterise the satisfiability and the equivalence classes thereof. For a deeper look into the motivation for the study in this note, the reader is referred to [1, 2, 3, 4, 5]; for more information on the application of the results described in this note, the reader is referred to [5].

Definition 1.1 (Time)**.** Let $\mathbb{D} \leq \mathbb{R}$ be an additive subgroup of the real numbers that is closed under the infimum operation on bounded nonempty subsets. We model *time* as the nonnegative part of this group: $\mathbb{T} \triangleq \{t \in \mathbb{D} \mid t \geq 0\}$.

Remark 1.2. The reader may verify that $\mathbb{D}$ is either $\mathbb{R}$ (corresponding to continuous modelling of time), or cyclic (corresponding to discrete modelling of time) and hence isomorphic to $\mathbb{Z}$. In turn, $\mathbb{T}$ is either $\mathbb{R}_{\geq 0}$ (the nonnegative real numbers), or isomorphic to $\mathbb{N} \cup \{0\}$, respectively.

Definition 1.3 (Time Difference Bounds)**.** We define $\Delta = \mathbb{D} \cup \{-\infty, \infty\}$.

We now turn to model the timely constraints imposed on the actions of the various agents.

Definition 1.4. Let I be a set. We denote the set of ordered pairs of distinct elements of I by $I^{\bar{2}} \triangleq \{(i, j) \in I^2 \mid i \neq j\}$.

Definition 1.5 (Satisfiability of Timely Constraints)**.**

1. We call a pair (I, δ) a *timely specification* if I is a set (of arbitrary cardinality) and if δ is a function $\delta : I^{\bar{2}} \to \Delta$. We call I the set of *actions* and call δ a *timely constraint*.

2. Let (I, δ) be a timely specification. We say that a function $t : I \to \mathbb{T}$, specifying a time for every action, *satisfies* δ (i.e. satisfies (I, δ)), if t satisfies $t(j) \leq t(i) + \delta(i, j)$ for every $(i, j) \in I^{\bar{2}}$. We denote the set of all functions satisfying δ by $T(\delta)$. If $T(\delta) \neq \emptyset$, we say that δ is *satisfiable*; otherwise, we say that δ is *unsatisfiable*.

Remark 1.6. Obviously, δ is unsatisfiable unless $\delta > -\infty$ (in every coordinate). Nonetheless, we still allow δ to take on the value of $-\infty$ for some or all pairs of actions, for technical reasons that become apparent when we define a canonisation operation on timely constraints in Section 2.

Observation 1.7. *Let (I, δ) be a timely specification. By Definition 1.5,*

- $-\delta(j, i) \leq t(j) - t(i) \leq \delta(i, j)$, *for every $t \in T(\delta)$ and every $(i, j) \in I^{\bar{2}}$.*

- *Let $t : I \to \mathbb{T}$. If $t \in T(\delta)$, then $t + c \in T(\delta)$ as well, for every $c \in \mathbb{T}$, as well as for every other $c \in \mathbb{D}$ s.t. $t + c \geq 0$.*

- *T is order-preserving: Let $\delta' : I^{\bar{2}} \to \Delta$. If $\delta' \leq \delta$, then $T(\delta') \subseteq T(\delta)$.*

In the rest of this note, we embark on a graph-theoretic discussion with the aim of analysing the above definitions. In Section 2, we provide a more tangible characterisation for satisfiability of a timely constraint, define a "nicely-behaved" canonical representative $\hat{\delta}$ for each set of timely constraints that share

the same image under the mapping T, and constructively build a minimal satisfying function for each such set. In Section 3, we show that the quotient map $\hat{\delta} \mapsto T(\delta)$, for satisfiable timely constraints, is an order-embedding (i.e. it is both order-preserving and order-reflecting; thus it is also one-to-one), and apply this result to further characterise the canonical form $\hat{\delta}$.

2. Satisfiability

As a first step toward analysing the satisfiability of a timely constraint, we define a canonisation operation on timely constraints, which preserves the set of satisfying functions. In order to define the canonical form of a timely constraint δ, we consider δ as a weight function on the edges of a directed graph on I.

Definition 2.1 (Associated Graph). Let (I, δ) be a timely specification.

1. We define the weighted directed graph of δ as $G_\delta \triangleq (I, E_\delta, \delta|_{E_\delta})$, where the set of edges is defined as $E_\delta \triangleq \{(i, j) \in I^2 \mid \delta(i, j) < \infty\}$.

2. We denote the set of paths in G_δ by $\mathcal{P}(G_\delta)$. We denote the length of a path $\bar{p} = (p_m)_{m=1}^n \in \mathcal{P}(G_\delta)$ by $L_{G_\delta}(\bar{p}) \triangleq \sum_{m=1}^{n-1} \delta(p_m, p_{m+1}) < \infty$.

Definition 2.2 (Canonical Form). Let (I, δ) be a timely specification. We define the *canonical form* $\hat{\delta}$ of δ as the distance function on G_δ. By slight abuse of notation, we allow ourselves to write $\hat{\delta}$ instead of $\hat{\delta}|_{I^2}$ on some occasions below.

Observation 2.3 (Elementary Properties of the Canonical Form). *Let (I, δ) be a timely specification. By Definition 2.2, we obtain the following properties of $\hat{\delta}$:*

- *$\forall i \in I : \hat{\delta}(i, i) \in \{0, -\infty\}$. (Thus, by Observation 1.7, for satisfiable δ we obtain $\hat{\delta}|_{\{(i,i)|i \in I\}} \equiv 0$.) Furthermore, $\hat{\delta}(i, i) = -\infty$ iff i is a vertex along a negative cycle in G_δ.*

- *Idempotence: $\hat{\hat{\delta}} = \hat{\delta}$.*

- *Minimality[1]: $\hat{\delta} \leq \delta$.*

- *Triangle inequality: $\forall i, j, k \in I : \hat{\delta}(i, k) \leq \hat{\delta}(i, j) + \hat{\delta}(j, k)$.*

- *Equivalence: $T(\hat{\delta}) = T(\delta)$. ($\supseteq$: by definition of $\hat{\delta}$. $\subseteq$: by minimality and by Observation 1.7 (monotonicity of T).)*

- *Order preservation: Let $\delta' : I^{\bar{2}} \to \Delta$. If $\delta' \leq \delta$, then $\widehat{\delta'} \leq \hat{\delta}$.*

We are now ready to characterise the satisfiable timely constraints on a set I. The first part of the following lemma performs this task, while its second part constructively shows that for every satisfiable δ, there exists a satisfying function that is minimal in every coordinate — a result that is of essence

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[1]This name is justified in Section 3.

in order to optimally solve a large class of naturally-occurring coordination problems [1, 2, 3, 4, 5].

Lemma 2.4 (Satisfiability Criterion). *Let (I, δ) be a timely specification.*

1. *δ is satisfiable iff $\hat{\delta}|_{\{i\} \times I}$ is bounded from below for each $i \in I$.*

2. *If δ is satisfiable, then $i \mapsto -\inf(\hat{\delta}|_{\{i\} \times I})$ satisfies δ, and is minimal in every coordinate with regard to this property.[2]*

Proof. We first prove that if δ is satisfiable, then for every $t \in T(\delta)$ and for every $i \in I$, we have $t(i) \geq -\inf(\hat{\delta}|_{\{i\} \times I})$. This implies the first direction ("$\Rightarrow$") of the first part of the lemma, and the minimality in the second part of the lemma.

Assume that δ is satisfiable and let $t \in T(\delta)$. By Observation 2.3 (equivalence), $t \in T(\hat{\delta})$ as well. Let $i \in I$. By definition of satisfiability, we obtain

$$\forall j \in I \setminus \{i\} : \hat{\delta}(i, j) \geq t(j) - t(i) \geq 0 - t(i) = -t(i).$$

By Observation 2.3, $\hat{\delta}(i, i) = 0 \geq -t(i)$. Thus, we have $\hat{\delta}|_{\{i\} \times I} \geq -t(i)$, and hence $\inf(\hat{\delta}|_{\{i\} \times I}) \geq -t(i)$, completing this part of the proof.

We now prove that if $\hat{\delta}|_{\{i\} \times I}$ is bounded from below for each $i \in I$, then the function defined in the second part of the lemma indeed satisfies δ. This completes the proof of both parts of the lemma.

Define $t : I \to \mathbb{T}$ by $i \mapsto -\inf(\hat{\delta}|_{\{i\} \times I}) < \infty$. For every $i \in I$, by Observation 2.3, $\hat{\delta}(i, i) \leq 0$, and therefore $t(i) \geq 0$. Thus, t is well defined. Let $(i, j) \in I^2$ and let $\bar{p} = (p_m)_{m=1}^n \in \mathcal{P}(G_\delta)$ s.t. $p_1 = j$. Define $p_0 \triangleq i$. Note that

$$\inf(\hat{\delta}|_{\{i\} \times I}) \leq L_{G_\delta}\big((p_m)_{m=0}^n\big) = \delta(i, j) + L_{G_\delta}\big((p_m)_{m=1}^n\big).$$

By taking the infima of both sides over all $\bar{p} \in \mathcal{P}(G_\delta)$ s.t. $p_1 = j$, we obtain $\inf(\hat{\delta}|_{\{i\} \times I}) \leq \delta(i, j) + \inf(\hat{\delta}|_{\{j\} \times I})$. Thus, $t(j) \leq t(i) + \delta(i, j)$, as required. $\square$

Example 2.5. Let (I, δ) be a timely specification. If $\delta \geq 0$ (i.e. no lower bound is given on the proximity of any pair of actions), then δ is satisfiable, and its minimal satisfying function is $t \equiv 0$.

For the case in which I is finite, the first part of Lemma 2.4 yields the following, even more tangible, satisfiability criterion.

Corollary 2.6 (Satisfiability Criterion — Finite Case). *Let (I, δ) be a timely specification s.t. $|I| < \infty$ and $\delta > -\infty$. δ is satisfiable iff G_δ contains no negative cycles.*

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[2]A quick glance at the formulation of this minimal satisfying function may raise a suspicion that perhaps it would have been more natural to define δ as the negation (in every coordinate) of the definition we have given. While it is indeed possible to define δ this way, and while doing so would have indeed given a more natural definition of the minimal satisfying function, it would have also required us to work with greatest path lengths instead of distances, with a reverse triangle inequality and with order-reversing monotonicity, which may seem less natural.

We conclude this section by showing, by means of a simple example, that the finiteness condition in Corollary 2.6 cannot be dropped.

Example 2.7. Set $I \triangleq \mathbb{N}$. Define $\delta : I^{\bar{2}} \to \Delta$ by $\delta(1, n) \triangleq -n$ for every $n \in \mathbb{N} \setminus \{1\}$, and ∞ in all other coordinates. It is easy to see that δ is unimplementable (either directly: what would $t(1)$ be?; or using Lemma 2.4, as $\delta|_{\{1\} \times I}$ is unbounded from below, and therefore neither is $\hat{\delta}|_{\{1\} \times I}$), even though G_δ contains no negative cycles. (In fact, G_δ is a star, and thus contains no cycles at all.)

3. Uniqueness of the Canonical Form

We now prove a uniqueness property one may expect from the canonical form defined above, namely that the equivalence classes of satisfiable timely constraints, under the equivalence relation $\delta_1 \sim \delta_2 \Leftrightarrow T(\delta_1) = T(\delta_2)$, are in one-to-one correspondence with canonical forms. Furthermore, we show that the quotient map $T/{\sim}$, mapping canonical forms (as representatives of equivalence classes) to sets of satisfying functions, is an order-embedding. We use these results to deduce additional, equivalent, definitions for the canonical form, each shedding a different light thereon. At the heart of all the results in this section lies the following lemma, constructively demonstrating that each coordinate $\hat{\delta}(i, j)$ of the canonical form $\hat{\delta}$ of a timely constraint δ captures the upper bound imposed by δ on $t(j) - t(i)$ in the tightest manner possible.

Lemma 3.1 (Attainability of Canonical Constraints)**.** *Let (I, δ) be a timely specification s.t. δ is satisfiable, and let $\tilde{\imath}, \tilde{\jmath} \in I$.*

1. *If $\hat{\delta}(\tilde{\imath}, \tilde{\jmath}) < \infty$, then there exists $t \in T(\delta)$ satisfying $t(\tilde{\jmath}) - t(\tilde{\imath}) = \hat{\delta}(\tilde{\imath}, \tilde{\jmath})$.*

2. *If $\hat{\delta}(\tilde{\imath}, \tilde{\jmath}) = \infty$, then for every $K \in \mathbb{T}$, there exists $t \in T(\delta)$ satisfying $t(\tilde{\jmath}) - t(\tilde{\imath}) \geq K$.*

Proof. By Lemma 2.4, $\forall i \in I : \exists d_i \in \mathbb{T} : \hat{\delta}|_{\{i\} \times I} \geq -d_i$. (For the time being, we may choose $(-d_i)_{i \in I}$ to be the infima of the respective restrictions of $\hat{\delta}$.) We define $\delta' : I^{\bar{2}} \to \Delta \setminus \{\infty\}$ by

$$\forall i, j \in I^{\bar{2}} : \delta'(i, j) = \begin{cases} \delta(i, j) & \delta(i, j) < \infty \\ d_j & \delta(i, j) = \infty. \end{cases}$$

As $\delta' \leq \delta$, by Observation 1.7 (monotonicity of T) it is enough to find $t \in T(\delta')$ that meets the conditions of the lemma. By Observation 2.3 (equivalence), this is equivalent to finding $t \in T(\widehat{\delta'})$ that meets the conditions of the lemma.

We start by showing that $\forall i \in I : \widehat{\delta'}|_{\{i\} \times I} \geq -d_i$. Let $i \in I$ and let $\bar{p} = (p_m)_{m=1}^n \in \mathcal{P}(G_{\delta'})$ s.t. $p_1 = i$. Set $l = \left| \left\{ k \in [n-1] \mid \delta(p_k, p_{k+1}) = \infty \right\} \right|$, where $[n-1] \triangleq \{1, \ldots, n-1\}$; l is the number of "new" edges in $\bar{p}$, which do not exist in G_δ. We show, by induction on l, that $L_{G_{\delta'}}(\bar{p}) \geq -d_i$.

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Base: If $l = 0$, then $L_{G_{\delta'}}(\bar{p}) = L_{G_{\delta}}(\bar{p}) \geq -d_i$.

Induction step: Assume that $l \geq 1$. Let $k \in [n-1]$ be maximal such that (p_k, p_{k+1}) is a "new" edge (i.e. $\delta(p_k, p_{k+1}) = \infty$). By definition of δ', we have $\delta'(p_k, p_{k+1}) = d_{p_{k+1}}$. Thus, by the induction hypothesis and by definition of $d_{p_{k+1}}$, we obtain

$$
\begin{aligned}
L_{G_{\delta'}}(\bar{p}) = L_{G_{\delta'}}\big((p_m)_{m=1}^{k}\big) \quad &+ \delta'(p_k, p_{k+1}) \quad + L_{G_{\delta}}\big((p_m)_{m=k+1}^{n}\big) \quad \geq \\
\geq - d_i \quad\quad &+ d_{p_{k+1}} \quad\quad - d_{p_{k+1}} \quad\quad\quad = - d_i,
\end{aligned}
$$

and the proof by induction is complete. In particular, we conclude that $\widehat{\delta'} > -\infty$. (Recall that by definition, also $\widehat{\delta'} \leq \delta' < \infty$.)

We claim that $t \triangleq d_{\tilde{\imath}} + \widehat{\delta'}(\tilde{\imath}, \cdot) \geq 0$ satisfies $\widehat{\delta'}$ (and hence also satisfies δ' and δ). Indeed, by Observation 2.3 (triangle inequality), for every $(j, k) \in I^2$ we have

$$
t(k) = d_{\tilde{\imath}} + \widehat{\delta'}(\tilde{\imath}, k) \leq d_{\tilde{\imath}} + \widehat{\delta'}(\tilde{\imath}, j) + \widehat{\delta'}(j, k) = t(j) + \widehat{\delta'}(j, k).
$$

If $\hat{\delta}(\tilde{\imath}, \tilde{\jmath}) < \infty$, we define $K \triangleq \hat{\delta}(\tilde{\imath}, \tilde{\jmath})$; otherwise, let $K \in \mathbb{T}$ be arbitrarily large as in the second part of the lemma. As δ' is satisfiable, by Observation 2.3 we obtain $\widehat{\delta'}(\tilde{\imath}, \tilde{\imath}) = 0$. Therefore,

$$
t(\tilde{\jmath}) - t(\tilde{\imath}) = \big(d_{\tilde{\imath}} + \widehat{\delta'}(\tilde{\imath}, \tilde{\jmath})\big) - \big(d_{\tilde{\imath}} + \widehat{\delta'}(\tilde{\imath}, \tilde{\imath})\big) = \widehat{\delta'}(\tilde{\imath}, \tilde{\jmath}).
$$

Thus, if $\widehat{\delta'}(\tilde{\imath}, \tilde{\jmath}) \geq K$, then the proof is complete. (For the case in which $\hat{\delta}(\tilde{\imath}, \tilde{\jmath}) < \infty$, we obtain $\widehat{\delta'}(\tilde{\imath}, \tilde{\jmath}) \leq K$ by Observation 2.3 (monotonicity), since $\delta' \leq \delta$.)

Otherwise, set $d \triangleq K - \widehat{\delta'}(\tilde{\imath}, \tilde{\jmath}) > 0$, and for every $i \in I$ define $d'_i \triangleq d_i + d > d_i$. Therefore, $-d'_i < -d_i \leq \hat{\delta}|_{\{i\} \times I}$ for every $i \in I$. Denote by δ'' the function constructed from δ in the same way in which δ' was constructed from it, but using the lower bounds $(-d'_i)_{i \in I}$ rather than $(-d_i)_{i \in I}$. As explained above, in order to prove that $d'_{\tilde{\imath}} + \widehat{\delta''}(\tilde{\imath}, \cdot)$ satisfies the conditions of the lemma, it is enough to show that $\widehat{\delta''}(\tilde{\imath}, \tilde{\jmath}) \geq K$. Let $\bar{p} = (p_m)_{m=1}^{n} \in \mathcal{P}(G_{\delta''})$ s.t. $p_1 = \tilde{\imath}$ and $p_n = \tilde{\jmath}$. If $\forall k \in [n-1] : \delta(p_k, p_{k+1}) < \infty$, then $L_{G_{\delta''}}(\bar{p}) = L_{G_{\delta}}(\bar{p}) \geq \hat{\delta}(\tilde{\imath}, \tilde{\jmath}) \geq K$. Otherwise,

$$
\begin{aligned}
L_{G_{\delta''}}(\bar{p}) = \quad & \text{by definitions of } \delta' \text{ and } \delta'' \\
= L_{G_{\delta'}}(\bar{p}) + d \cdot \Big|\big\{k \in [n-1] \mid \delta(p_k, p_{k+1}) = \infty\big\}\Big| \geq \quad & \text{as this set is non-empty} \\
\geq L_{G_{\delta'}}(\bar{p}) + d \geq \quad & \text{by definition of } \widehat{\delta'} \\
\geq \widehat{\delta'}(\tilde{\imath}, \tilde{\jmath}) + d = \quad & \text{by definition of } K \\
= K.
\end{aligned}
$$

Either way, the proof is complete. $\qquad\square$

While there exist unsatisfiable timely constraints whose canonical forms differ (due to G_δ not necessarily being strongly connected and to I no necessarily

being finite), we now conclude, using Lemma 3.1, that for satisfiable timely constraints, the map $\hat{\delta} \mapsto T(\delta)$, from the canonical form of a satisfiable timely constraint δ to the set of functions satisfying δ (this map is well defined by Observation 2.3 — equivalence), is an order-embedding (and thus, in particular, also one-to-one). This gives way to the use of the canonical form as an efficient tool for classifying and sorting timely constraints according to their "strictness".

Corollary 3.2 ($\hat{\delta} \mapsto T(\delta)$ is an Order-Embedding). *Let I be a set and let $\delta_1, \delta_2 : I^{\bar{2}} \to \Delta$ s.t. δ_1 is satisfiable. $\hat{\delta}_1 \leq \hat{\delta}_2$ iff $T(\delta_1) \subseteq T(\delta_2)$.*

Proof. $\Rightarrow$: Assume that $\hat{\delta}_1 \leq \hat{\delta}_2$. By Observation 1.7 (monotonicity of T) and by Observation 2.3 (equivalence), we have $T(\delta_1) = T(\hat{\delta}_1) \subseteq T(\hat{\delta}_2) = T(\delta_2)$.

$\Leftarrow$: Assume that $\hat{\delta}_1 \not\leq \hat{\delta}_2$. Thus, there exist $\tilde{\imath}, \tilde{\jmath} \in I$ s.t. $\hat{\delta}_1(\tilde{\imath}, \tilde{\jmath}) > \hat{\delta}_2(\tilde{\imath}, \tilde{\jmath})$. If $\hat{\delta}_1(\tilde{\imath}, \tilde{\jmath}) < \infty$, then by Lemma 3.1 there exists $t \in T(\delta_1)$ s.t. $t(\tilde{\jmath}) - t(\tilde{\imath}) = \hat{\delta}_1(\tilde{\imath}, \tilde{\jmath}) > \hat{\delta}_2(\tilde{\imath}, \tilde{\jmath})$, and thus $t \in T(\delta_1) \setminus T(\delta_2)$, and the proof is complete.

If $\hat{\delta}_1(\tilde{\imath}, \tilde{\jmath}) = \infty$, then $\hat{\delta}_2(\tilde{\imath}, \tilde{\jmath}) < \infty$ and thus there exists $K \in \mathbb{T}$ s.t. $K > \hat{\delta}_2(\tilde{\imath}, \tilde{\jmath})$. Similarly to the proof of the previous case, by Lemma 3.1 there exists $t \in T(\delta_1)$ s.t. $t(\tilde{\jmath}) - t(\tilde{\imath}) \geq K > \hat{\delta}_2(\tilde{\imath}, \tilde{\jmath})$. Once again, we obtain that $t \in T(\delta_1) \setminus T(\delta_2)$, and the proof is complete. $\square$

Corollary 3.3 (Uniqueness of the Canonical Form). *Let I be a set and let $\delta_1, \delta_2 : I^{\bar{2}} \to \Delta$ s.t. at least one of them is satisfiable. $\hat{\delta}_1 = \hat{\delta}_2$ iff $T(\delta_1) = T(\delta_2)$.*

Proof. $\Rightarrow$: Assume that $\hat{\delta}_1 = \hat{\delta}_2$. By Observation 2.3 (equivalence), we have $T(\delta_1) = T(\hat{\delta}_1) = T(\hat{\delta}_2) = T(\delta_2)$.

$\Leftarrow$: Assume that $T(\delta_1) = T(\delta_2)$. Thus, since at least one of δ_1, δ_2 is satisfiable, they both are. To complete the proof, we apply Corollary 3.2 to $T(\delta_1) \subseteq T(\delta_2)$ and to $T(\delta_2) \subseteq T(\delta_1)$. $\square$

The above discussion gives rise to two alternative definitions (or rather, characterisations) of the canonical form of satisfiable functions. The first one justifies the name of the minimality property from Observation 2.3, and stems from this property when combined with Corollary 3.3. The second one, which explicitly defines the inverse of the order-embedding $\hat{\delta} \mapsto T(\delta)$, stems directly from Lemma 3.1 and from the definition of satisfiability. These definitions, both non-constructive in nature (in contrast with Definition 2.2), showcase once more the fact that the canonical form indeed emerges naturally, and that choosing it to represent equivalence classes of timely constraints is not merely an artifact of its being possible to constructively define and efficient to calculate.

Corollary 3.4 (Characterizations of the Canonical Form). *Let (I, δ) be a timely specification s.t. δ is satisfiable.*

1. $\hat{\delta} = \min\{\delta' \in \Delta^{(I^{\bar{2}})} \mid T(\delta') = T(\delta)\}$.[3]

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[3]In particular, there exists a function in this set that is minimal in every coordinate, however this may be directly proven by means of a simpler argument.

2. $\forall i, j \in I : \hat{\delta}(i, j) = \sup\{t(j) - t(i) \mid t \in T(\delta)\}$.[4]

In fact, satisfiability of δ is not required in Corollary 3.4 if G_δ is strongly connected and if I is finite. Indeed, under these conditions, if δ is unsatisfiable, then $\hat{\delta} \equiv -\infty$, which coincides with both parts of this corollary when applied to any unsatisfiable δ.[5] This suggests modifying the definition of the canonical form of any unsatisfiable timely constraint to be $\hat{\delta} \triangleq -\infty$, in which case Corollaries 3.2, 3.3 and 3.4 apply to unsatisfiable timely constraints as well. As aesthetically-appealing as such a definition may be, however, we note that it renders the canonical form useless as a tool for checking the solvability of a timely constraint (as the question of solvability must be answered in order to compute the canonical form under this definition). Indeed, in this case checking the solvability of a timely constraint (and thus also computing its canonical form) still involves computing $\hat{\delta}$ as it is defined in Definition 2.2, and then applying the satisfiability criterion from Lemma 2.4 (and only then, if the timely constraint turns out to be unsatisfiable, amending its canonical form to equal $-\infty$ across all coordinates).

4. Discussion and Further Reading

The problems discussed in this note seem analytic in nature, perhaps more conventionally approached via geometric tools. Nonetheless, as we have seen, these problems give way to natural analysis by the somewhat unexpected use of graph theory, not only providing a gamut of powerful and insightful theoretical results (regardless of the cardinality of I), but also, when I is finite, reducing the problem of satisfiability of a timely constraint to that of negative-cycle detection, and the problems of finding minimal satisfying functions and of classifying and comparing timely constraints — to all-pairs shortest-path computation (where finding a single coordinate of a minimal satisfying function is reduced to single-source shortest-path computation). This allows us to harness the vast existing knowledge regarding these computational graph problems in order to efficiently analyse timely constraints. For example, for a single timely constraint, all these problems may be jointly solved via a single run of the Floyd-Warshall algorithm [6], in $O(|I|^3)$ time. Recently-discovered all-pairs shortest-path algorithms, such as Han's [7], provide even better asymptotic complexity. In the special cases in which $\delta \geq 0$[6] and/or G_δ is sparse (i.e. $\delta(i, j) = \infty$ for many $(i, j) \in I^2$), other well-known algorithms may be used to even further improve

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[4]If this set is bounded from above, then by Lemma 3.1, it attains its supremum.

[5]When G_δ is not strongly connected, then the coordinates i for which $\inf(\hat{\delta}|_{\{i\} \times I})$ assumes the value of $-\infty$ (if $|I| < \infty$, these are the coordinates for which $\hat{\delta}(i, i) = -\infty$) indicate the connected component(s) of G_δ in which the "reason(s)" for unsatisfiability (e.g. the negative cycle(s), if $|I| < \infty$) lie(s).

[6]In this case, δ is always satisfiable, and finding a minimal satisfying function is trivial (see Example 2.5); however, classifying δ and comparing it to other timely constraints may still be of interest.

the running time; the interested reader is referred to the notes concluding [8, Chapter 25].

It should be noted that under many distributed models, the graph G_δ, associated with a timely constraint δ, plays an even more pivotal role in the study of timely coordination than seen in this note. For example, its strongly-connected components are instrumental in characterising the communication channels (between the agents corresponding to the various actions) required for solving timely-coordinated response problems associated with δ. (For example, all the coordination problems introduced in [1, 2, 3, 4, 9] may readily be reformulated using an appropriate δ and analysed in this way.) For the details, which are beyond the scope of this note, the reader is referred to [5].

5. Acknowledgements

References

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